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Goldstone singularities and critical behaviour in isotropic systems

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Abstract. The critical behaviour along the coexistence curve of isotropic ferromagnets is characterized by two diverging correlation lengths since longitudinal and transverse fluctuations of the order parameter become critical. Therefore, a critical theory valid for the whole phase diagram has to incorporate the crossover between a region characterized by one correlation length to another region where two different correlation lengths dominate. Treating this crossover problem entirely within the framework of the trajectory integral method, I have calculated the equation of state and the susceptibilities. It is shown that coexistence behaviour is governed by a coexistence fixed point which is characterized by the vanishing interaction among the Goldstone modes. Taking the coupling of the critical transverse modes to the longitudinal modes into account, it is shown that coexistence behaviour is characterized by a Fisher renormalization. This central result sheds new light on the symmetry-broken phase in isotropic systems. The asymptotic forms of the equation of state and the susceptibilities at the coexistence curve are expressed by the specific heat exponent $\alpha_t = \frac{1}{2}$ for $d \geqslant 3$ and for all spin dimensions. These results are in perfect agreement with the nonlinear σ-model. The crossover from coexistence curve behaviour to ordinary critical behaviour is calculated for the equation of state and for the susceptibilities to $O(\varepsilon)$ ($\varepsilon = 4 - d$).

1. Introduction

The critical behaviour of isotropic n-component systems has attracted theoretical interest for many years. A rather complete understanding of ordinary critical behaviour in the symmetric phase has been obtained by renormalization group methods. The critical behaviour in the symmetry-broken phase is more difficult to understand since two correlation lengths diverge at the coexistence curve. These correlation lengths correspond to fluctuations longitudinal and transverse to the direction of magnetization. The latter are also called Goldstone modes. Their mass vanishes and the transverse susceptibility diverges at the coexistence curve; this induces non-analytic behaviour into the equation of state and the longitudinal susceptibility near the coexistence curve.

In the language of critical phenomena the coexistence curve is a line of critical points where the correlation length of the transverse fluctuations diverges. It terminates at the ordinary critical point $T = T_c$, $M = 0$ which is a kind of bicritical point because longitudinal and transverse fluctuations become critical. Thus, a renormalization group

(RG) theory of coexistence has to describe the crossover between critical phenomena characterized by one correlation length and critical phenomena with two independent correlation lengths. This point of view is implicit in most papers on this subject.

The behaviour of isotropic ferromagnets near the coexistence curve has been studied in the framework of spin wave theory by Vaks *et al* [1, 2] who extended the Holstein-Primakoff formalism to calculate the magnetization and the specific heat of Heisenberg ferromagnets with arbitrary spin S. The main result was that the longitudinal susceptibility χ_L diverges like $H^{-1/2}$ and that the coupling of the longitudinal and the transverse modes, as well as the coupling among the transverse modes, vanishes for small wave vectors $\boldsymbol{q}$ at the coexistence curve. The validity of these results was restricted to temperatures outside the critical region since these works were essentially mean field theories.

The renormalization group (RG) has been applied to Heisenberg ferromagnets in a considerable number of papers using many different technical variants of the RG. There have been essentially two different field theoretic models to describe the critical behaviour of Heisenberg ferromagnets: the S^4-model and the nonlinear σ-model.

The first works were done on the S^4-model by Brezin *et al* [3]. They calculated the susceptibilities and the equation of state in an ε-expansion around $d_c = 4$ ($\varepsilon = 4 - d$). They obtained corrections to mean field theory as powers of $\varepsilon \ln r_T$ where r_T is the mass of the Goldstone modes which vanishes at the coexistence curve. Unfortunately, the renormalization group did not furnish the functional forms of the equation of state and the susceptibilities to which these logarithms could be exponentiated.

Historically, further investigations were performed on the nonlinear σ-model. This model focuses attention directly on the Goldstone modes. It was obtained by a low-temperature expansion of the partition function of the Heisenberg Hamiltonian. Brezin and Wallace [4] have shown within a $1/n$ expansion that the nonlinear σ-model and the S^4-model should have the same critical behaviour. Further analysis by Brezin and Zinn-Justin [5] showed that the behaviour of the nonlinear σ-model is governed by two fixed points. A trivial one which governs coexistence behaviour and a non-trivial one which controls the critical point. Thus, one had a scheme which incorporates both regimes and the crossover between them. However, the necessity to perform a low-temperature expansion and an expansion around $d_c = 2$ was clearly a drawback of the nonlinear model.

The coexistence behaviour in the S^4-model was further studied by techniques which combine the RG with resummation procedures. Rudnick and Nelson [6] have proposed such a technique, the trajectory integral method, to treat crossover problems and to calculate exponentiated critical singularities. The idea was to renormalize the critical system until the fluctuations are small so that Landau theory plus fluctuation corrections could be applied. However, this method failed when applied to the coexistence curve since longitudinal and transverse fluctuations independently become critical. Therefore, Nelson [7] applied a parquet graph summation to obtain exponentiated results which incorporate the Goldstone singularities.

According to Schäfer and Horner [8] this procedure did not treat the longitudinal fluctuations in the adequate way. Moreover, they claimed that a proper evaluation of the perturbation expansion necessitates working with a momentum-dependent four-spin coupling of the transverse modes. Schäfer and Horner proposed a resummation of renormalized perturbation theory without using recursion relations. Their work was based on the results of spin wave theory and of the nonlinear σ-model that the interaction between the transverse modes vanishes at the coexistence curve. They obtained exponentiated results for the susceptibility and the equation of state valid in the whole critical region.

Such an exponentiation was also obtained by Nicoll and Chang [9] who proposed a new infinitesimal formulation of the RG. In their method the Goldstone modes did not deserve any special treatment. They applied this method to calculate the free energy and the equation of state of isotropic Heisenberg ferromagnets in exponentiated form.

Lawrie [10] applied another technical variant of the RG, designed for bicritical crossover, to the coexistence curve of isotropic ferromagnets. Besides the well known critical fixed point, he found a coexistence fixed point which governs Goldstone behaviour and was able to describe the crossover between these fixed points.

To summarize, the critical behaviour at the critical point and along the coexistence curve seems to be quite well understood. The S^4-model, as well as the nonlinear σ-model, leads to a two-fixed-point scenario; however, the results obtained in the expansions around $d_c = 2$ and $d_c = 4$ respectively cannot be compared. The results obtained by RG methods [3-10] are in accord with the classical theory and agree with each other at least in essential limits such as the spherical limit. However, apart from some limiting cases the detailed results are hard to compare due to their mathematical complexity.

In this paper I show that the aspect of two diverging correlation lengths is crucial to an understanding of coexistence behaviour. This aspect has not been incorporated carefully enough in previous works [3-10]: on the one hand the nonlinear σ-model strips off the longitudinal fluctuations and focuses on the Goldstone modes directly. On the other hand, RG studies of the S^4-model did not adequately treat the interaction between both types of fluctuations.

In this paper I use the trajectory integral method to calculate the equation of state and the susceptibilities in the S^4-model. I point out that a different technical variant of the RG such as the differential generator [9] should lead to the same results. The idea is a simple but necessary extension of the ordinary renormalization scheme: the renormalization of the complete system is performed until the longitudinal fluctuations are non-critical; subsequently the longitudinal modes are eliminated in the partition function by integration of the longitudinal modes. This leads to an effective Hamiltonian for the transverse modes with renormalized coupling parameters; this effective Goldstone system is further renormalized until the transverse fluctuations become non-critical too.

The results of this approach provide a link between the results of the nonlinear σ-model and the S^4-model. The basic results of the nonlinear σ-model concerning the longitudinal susceptibility and the equation of state are reproduced. As the central result I show that the Goldstone behaviour appears as an excellent example for Fisher renormalization by hidden variables [11]. The physical origin of the Fisher renormalization is that the transverse modes are still critical but coupled to the longitudinal modes at the coexistence curve. The interaction among the transverse modes is shown to vanish at the coexistence curve. As a result of both effects Fisher-renormalized Gaussian behaviour dominates at the coexistence curve.

In section 2 the trajectory integral method is briefly explained. The full S^4 Hamiltonian is renormalized until the longitudinal modes are noncritical. In section 3 this partly renormalized S^4 Hamiltonian leads to a Hamiltonian for the $(n-1)$-dimensional Goldstone system. This Hamiltonian is further renormalized until the Goldstone modes are non-critical too. The matching conditions are evaluated in section 4 and the susceptibilities and the equation of state are derived in exponentiated scaling form.

Section 5 focuses on the coexistence curve and derives the coexistence curve singularities from the general equations of section 4. The asymptotic form for the susceptibilities and the equation of state are calculated and discussed.

2. The method

The starting point is the usual Ginzburg-Landau-Wilson Hamiltonian in its O(n)-symmetric form given by

$$\mathscr{H} = -\tfrac{1}{2}\int_q \boldsymbol{S}_q\cdot\boldsymbol{S}_{-q}(r+q^2) - u\int_{q_1}\int_{q_2}\int_{q_3}\boldsymbol{S}_{q_1}\cdot\boldsymbol{S}_{q_2}\cdot\boldsymbol{S}_{q_3}\cdot\boldsymbol{S}_{-(q_1+q_2+q_3)} + HS_0^1 \tag{2.1}$$

with n-component spins $\boldsymbol{S}$ and a cutoff Λ in the momentum integrals. The coupling parameters are $r\sim t=(T-T_c)/T_c$ and the S^4 coupling u. The latter is usually assumed to be independent of the temperature and the magnetic field in the critical region. H is the internal magnetic field which couples to the $q=0$-component S_0^1 in the 1-direction. The free energy density is given by

$$F = -\frac{1}{V}\ln\int_S \exp\mathscr{H}(\boldsymbol{S}) \tag{2.2}$$

where V is the volume of the system. One may conveniently introduce the shifted spin variables σ_0^1 in the usual way [3, 6, 7]

$$S_0^1 = M+\sigma_0^1 \qquad S_q^1=\sigma_q^1 \qquad (q\neq 0) \tag{2.3}$$

where M is the magnetization given by the vanishing expectation value $\langle\sigma_0^1\rangle=0$. The shift (2.3) leads to the Hamiltonian, which reads in short notation,

$$\begin{aligned}\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp) = &-\tfrac{1}{2}r_L\sigma^2-\tfrac{1}{2}r_T|\boldsymbol{S}_\perp|^2-w_1\sigma|\boldsymbol{S}_\perp|^2-w_2\sigma^3\\ &-u_1|\boldsymbol{S}_\perp|^4-u_2\sigma^4-u_3\sigma^2|\boldsymbol{S}_\perp|^2+\tilde{H}\sigma_0^1\end{aligned} \tag{2.4}$$

with the coupling parameters

$$\begin{aligned} &r_L = r+12uM^2 \qquad r_T=r+4uM^2\\ &w_1=w_2=4uM\\ &u_1=u_2=u \qquad u_3=2u\\ &\tilde{H}=H-rM-4uM^3.\end{aligned} \tag{2.5}$$

The idea of the trajectory integral method is to renormalize a system until it is non-critical. During the renormalization one sums up the regular parts of the free energy which are produced at each renormalization step [6, 7]. Applying the trajectory integral method to the Hamiltonian (2.4), one is lead to the free energy

$$F=\frac{r}{2}M^2+uM^4-HM+\int_0^{l^*}\tilde{G}_0(r_L(l),r_T(l),\ldots)\,\mathrm{e}^{-dl}\,\mathrm{d}l+\mathrm{e}^{-dl^*}\tilde{F}(l^*) \tag{2.6}$$

where the first terms result from the shift (2.3). The integration term is the integrated free energy resulting from the renormalization. The integration kernel $\tilde{G}_0$, which includes longitudinal as well as transverse fluctuations, is given by [6]

$$\tilde{G}_0(l)=\tfrac{1}{2}(n-1)K_4\ln(r_T(l)+1)+\tfrac{1}{2}K_4\ln(r_L(l)+1)+O(u(l)). \tag{2.7}$$

The last term in (2.6) is the matching term. $\tilde{F}(l^*)$ is the free energy of the renormalized system with the Hamiltonian $\tilde{\mathcal{H}}(l^*)$. The magnetic properties like the magnetization and the susceptibility are given by their simple scaling relations

$$M(r, u, H) = e^{-(1-\varepsilon/2)l^*} M(r(l^*), u(l^*), H(l^*)) \tag{2.8}$$

$$\chi(r, u, H) = e^{2l^*} \chi(r(l^*), u(l^*), H(l^*)). \tag{2.9}$$

The matching terms $\tilde{F}(l^*)$, $M(l^*)$ and $\chi(l^*)$ in (2.6), (2.8), (2.9) depend on the matching parameter l^*. According to the general idea of the trajectory integral method, l^* has to be chosen in such a way that the renormalized system $\tilde{\mathcal{H}}(l^*)$ is non-critical. The residual free energy, magnetization and susceptibility at l^* are calculated by Landau theory plus fluctuation corrections.

The renormalization of the Hamiltonian $\tilde{\mathcal{H}}$ (2.4) to calculate the shift and integral term in the free energy (2.6) can be performed in the usual way [7]. However, it is simpler not to renormalize the Hamiltonian $\tilde{\mathcal{H}}$ given by (2.4) but to renormalize the original Hamiltonian $\mathcal{H}$ given by (2.1) and then separate the spin mode S_0 into a fluctuating and a non-fluctuating part $S_0^1 = M(l^*) + \sigma_0^1$ in the renormalized Hamiltonian $\tilde{\mathcal{H}}(l^*)$. The advantage of this procedure is that one only has to solve the recursion relations of the S^4 model (2.1) and not those of the Hamiltonian (2.4). One easily finds that this interchange of shift and renormalization leads to the following solutions of the recursion relations:

$$\begin{aligned} &r_L(l^*) = r(l^*) + 12u(l^*)M^2(l^*) \qquad r_T(l^*) = r(l^*) + 4u(l^*)M^2(l^*) \\ &w_1(l^*) = w_2(l^*) = 4u(l^*)M(l^*) \\ &u_1(l^*) = u_2(l^*) = u(l^*) \qquad u_3(l^*) = 2u(l^*) \\ &\tilde{H}(l^*) = H(l^*) - r(l^*)M(l^*) - 4u(l^*)M^3(l^*) \end{aligned} \tag{2.10}$$

i.e. the solutions of the recursion relations in the symmetry-broken phase are related in a simple way to the solutions of the disordered phase. Small deviations of (2.10) from the solutions given by Nelson and Rudnick [6] stem from cutoff-dependent terms which can be neglected since they are irrelevant for the critical behaviour. This point has been discussed in a separate paper [12]. For completeness, I note the solutions of the recursion relations of the disordered phase [6, 12]

$$\begin{aligned} t(l) &= t\, e^{2l}\left(1 + \frac{Bu}{\varepsilon}(e^{\varepsilon l} - 1)\right)^{-A/B} \\ u(l) &= u\, e^{\varepsilon l}\left(1 + \frac{Bu}{\varepsilon}(e^{\varepsilon l} - 1)\right)^{-A/B} \end{aligned} \tag{2.11}$$

where $t(l) = r(l) + \frac{1}{2}A\Lambda^2 u(l)$. A and B are the usual parameters in the recursion relations of $\mathcal{H}$ (2.1) given by $A = 4(n+2)K_4$ and $B = 4(n+8)K_4$ [6]. The renormalization of the Hamiltonian (2.1) and the final shift of S_0 by $M(l^*)$ leads to an equivalent expression for the free energy (2.6)

$$F = \int_0^{l^*} G_0(r(l), \ldots)\, e^{-dl}\, dl + e^{-dl^*} \times \left[\frac{r(l^*)}{2} M^2(l^*) + u(l^*)M^4(l^*) - H(l^*)M(l^*)\right] + e^{-dl^*}\tilde{F}(l^*) \tag{2.12}$$

with the isotropic integration kernel $G_0(l)=\frac{1}{2}nK_4\ln(r(l)+1)$. Indeed, Nelson has obtained the same result for the free energy by a renormalization of the Hamiltonian (2.4). The first term in (2.12) is equal to the free energy of the system in the disordered phase. The last two terms are peculiar to the ordered phase.

The most important part of the free energy F (2.12) with respect to coexistence behaviour is the free energy $\tilde{F}(l^*)$ of the partly renormalized system $\tilde{\mathscr{H}}(l^*)$

$$\tilde{F}(l^*)=-\frac{1}{V(l^*)}\ln\int_{\sigma,S_\perp}\exp\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp;l^*). \tag{2.13}$$

Section 3 will be mainly concerned with the calculation of this part of the free energy. A simple mean field argument shows that the evaluation of $\tilde{F}(l^*)$ (2.13) by Landau theory plus fluctuation corrections fails at the coexistence curve: according to the original concept of the trajectory integral method the renormalized system is matched with a non-critical theory at some l^* where fluctuations are non-critical. If one chooses l^* in analogy to the symmetric phase such that the longitudinal fluctuations are non-critical ($r_{\rm L}(l^*)=O(1)$), then (2.10) leads to

$$M^2(l^*)=(-r(l^*)/4u(l^*)) \tag{2.14}$$

on the coexistence curve ($H=0$) neglecting fluctuation corrections. Inserting this mean field result for $M(l^*)$ into the solution of $r_{\rm T}(l^*)$ (2.10), one realizes that the renormalized transverse fluctuations are still fully critical at l^* near the coexistence curve:

$$r_{\rm T}(l^*)=r(l^*)+4u(l^*)M^2(l^*)\simeq 0. \tag{2.15}$$

Thus, one cannot apply Landau theory to the transverse fluctuations in $\tilde{\mathscr{H}}(l^*)$. In his paper Nelson applied the parquet graph summation to $\tilde{\mathscr{H}}(l^*)$ to sum up the most divergent Goldstone contributions to the susceptibilities and to the equation of state [7].

3. Renormalization of the Goldstone modes

The renormalization of the longitudinal modes has been performed in section 2 up to the matching point l^* where the longitudinal modes are non-critical ($r_{\rm L}(l^*)=O(1)$). We now proceed to renormalize the critical Goldstone modes contained in $\tilde{\mathscr{H}}(l^*)$ until they are non-critical at some matching point $\hat{l}$. To perform the renormalization of the Goldstone modes in $\tilde{\mathscr{H}}(l^*)$, it is necessary first to integrate the non-critical longitudinal modes in the free energy (2.13). To this end $\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp;l^*)$ is split into

$$\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp;l^*)=\tilde{\mathscr{H}}_0(\sigma)+\tilde{\mathscr{H}}_0(\boldsymbol{S}_\perp)+\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp) \tag{3.1}$$

with the Gaussian parts

$$\tilde{\mathscr{H}}_0(\sigma)=-\tfrac{1}{2}r_{\rm L}(l^*)\sigma^2 \qquad \tilde{\mathscr{H}}_0(\boldsymbol{S}_\perp)=-\tfrac{1}{2}r_{\rm T}(l^*)|\boldsymbol{S}_\perp|^2 \tag{3.2}$$

and the interaction

$$\begin{aligned}\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp)=&-w_1(l^*)\sigma|\boldsymbol{S}_\perp|^2-w_2(l^*)\sigma^3-u_1(l^*)|\boldsymbol{S}_\perp|^4-u_2(l^*)\sigma^4\\&-u_3(l^*)\sigma^2|\boldsymbol{S}_\perp|^2+\tilde{H}(l^*)\sigma\end{aligned} \tag{3.3}$$

using the short notation, without momentum-dependencies. The Feynman-graph expansion of $\tilde{F}(l^*)$ (2.13) with respect to the interaction $\tilde{\mathscr{H}}(\sigma, \boldsymbol{S}_\perp)$ leads to

$$\tilde{F}(l^*) = -\frac{1}{V(l^*)} \ln\left\{ \int_{S_\perp} \exp(\tilde{\mathscr{H}}_0(\boldsymbol{S}_\perp)) \exp(\langle e^{\tilde{\mathscr{H}}(\sigma,\boldsymbol{S}_\perp)} - 1\rangle_{0c}) \exp(-V(l^*)\Delta F_\sigma \right\}$$
$$= -\frac{1}{V(l^*)} \ln \int_{S_\perp} \exp(\hat{\mathscr{H}}(\boldsymbol{S}_\perp) - V(l^*)\Delta F_\sigma)$$
$$=: \hat{F} + \Delta F_\sigma. \tag{3.4}$$

ΔF_σ is the part of free energy resulting from the integration of the longitudinal modes in $\tilde{\mathscr{H}}(\sigma, \boldsymbol{S}_\perp; l^*)$ only. Since they are non-critical, ΔF_σ can be calculated by Landau theory plus leading fluctuation corrections:

$$\Delta F_\sigma = -\frac{\tilde{H}^2(l^*)}{2r_{\rm L}(l^*)} + \frac{K_4}{2}\int_p \ln(r_{\rm L}(l^*) + p^2) + 3\,\frac{w_2(l^*)\tilde{H}(l^*)}{r_{\rm L}(l^*)} \int_p \frac{1}{r_{\rm L}(l^*) + p^2}$$
$$+ {\rm O}(u_2(l^*), w_2^2(l^*)) \tag{3.5}$$

where $u_i(l^*) = w_1^2(l^*) = {\rm O}(u(l^*))$ is used, which follows from (2.10), (2.14). The first two terms in (3.5) are zero-order terms, and the last one is the leading fluctuation term from graph (a) in figure 1.

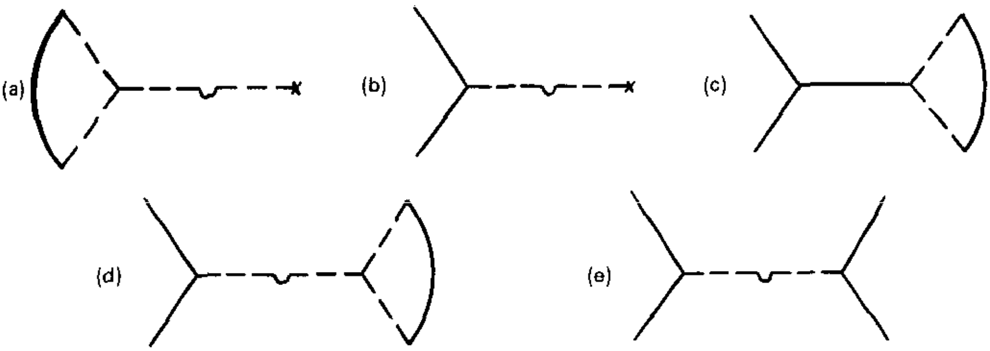

Figure 1. Graphs relevant for the Feynman graph expansion of the free energy $\tilde{F}(l^*)$ in equation (3.4). Graph (a) is the first non-trivial contribution to the integrated free energy ΔF_σ; graphs (b)–(d) contribute to the effective $S_\perp^2$ interaction and (e) contributes to the effective $S_\perp^4$ interaction in ${\rm O}(\varepsilon, u)$.

The effective Hamiltonian $\hat{\mathscr{H}}$ in (3.4) describes the critical Goldstone modes in the presence of non-critical longitudinal modes. It is derived in ${\rm O}(u(l^*), w^2(l^*))$ from the graphs (b)–(e) in figure 1. Graphs (b)–(d) contribute to the effective $|\boldsymbol{S}_\perp|^2$ vertex and (e) contributes to the effective $|\boldsymbol{S}_\perp|^4$ interaction. One obtains

$$\hat{\mathscr{H}}(\boldsymbol{S}_\perp) = -\tfrac{1}{2}\hat{r}|\boldsymbol{S}_\perp|^2 - \hat{u}|\boldsymbol{S}_\perp|^4 \tag{3.6}$$

with

$$\hat{r} = r_{\rm T}(l^*) + 2\,\frac{w_1(l^*)\tilde{H}(l^*)}{r_{\rm L}(l^*)} + 4u(l^*)G_{\rm L}(0) - 6\,\frac{w_1(l^*)w_2(l^*)}{r_{\rm L}(l^*)}\,G_{\rm L}(0) \tag{3.7}$$

and

$$\hat{u} = u(l^*) - \frac{w_1^2(l^*)}{2}\,G_{\rm L}(\boldsymbol{q}_1 + \boldsymbol{q}_2). \tag{3.8}$$

G_L is the integral over the longitudinal propagator at l^*:

$$G_L(\boldsymbol{q}) = \int_p \frac{1}{r_L(l^*) + (\boldsymbol{p}+\boldsymbol{q})^2} \tag{3.9}$$

where the integral is performed over the whole momentum sphere $0 \leq |\boldsymbol{p}| \leq \Lambda$. The resulting effective Hamiltonian $\hat{\mathscr{H}}$ depends only on the $(n-1)$-dimensional Goldstone modes. $\hat{\mathscr{H}}$ has the same symmetry as the original n-dimensional Hamiltonian $\mathscr{H}$ (2.1) before renormalization of the longitudinal modes. However, the coupling parameters have changed markedly due to the fact that the critical Goldstone modes are coupled to the non-critical longitudinal modes. This coupling is responsible for the Fisher renormalization which will become visible below when the new coupling parameters are expressed in terms of the original parameters of $\mathscr{H}$ (2.1). The effective Hamiltonian $\hat{\mathscr{H}}$ (3.6) of the Goldstone system may now be renormalized in the usual way. The trajectory integral method may be applied to integrate the free energy $\hat{F}$ (3.4) of this $(n-1)$-dimensional S^4-model. The recursion relations for $\hat{r}$ and $\hat{u}$ that have to be solved are those of the S^4-model with the spin dimension $n-1$. Thus, one obtains the same solutions (2.11) but for the effective temperature scaling field

$$\hat{t} = \hat{r} + \tfrac{1}{2}\hat{A}\Lambda^2\hat{u} \tag{3.10}$$

and the effective coupling parameter $\hat{u}$ with $\hat{A} = 4(n+1)K_4$ and $\hat{B} = 4(n+7)K_4$. $\hat{t}$ is called effective temperature since it is the relevant scaling field in the Goldstone system. However, it will be shown below that $\hat{t}$ is not a temperature difference like t but a distance from the coexistence curve in the M–T phase space. The free energy $\hat{F}$ of the Goldstone system is readily obtained as

$$\hat{F} = \int_0^{\hat{l}} \hat{G}_0(l)\, \mathrm{e}^{-dl}\, \mathrm{d}l + \mathrm{e}^{-d\hat{l}}\, \hat{F}(\hat{l}) \tag{3.11}$$

where $\hat{G}_0$ is the integration kernel of the $(n-1)$-dimensional Goldstone system. Obviously $\hat{l}$ is chosen in such a way that the renormalized Goldstone system is non-critical at $\hat{l}$, i.e. $\hat{r}(\hat{l}) = \mathrm{O}(1)$. Combining the result (3.11) and (3.4) with the free energy F (2.12) one obtains the complete free energy of the S^4-model (2.1) as

$$F = \int_0^{l^*} G_0(l)\, \mathrm{e}^{-dl}\, \mathrm{d}l + \mathrm{e}^{-dl^*}\left[\frac{r(l^*)}{2} M^2(l^*) + u(l^*)M^4(l^*) - H(l^*)M(l^*)\right]$$
$$+ \mathrm{e}^{-dl^*} \int_0^{\hat{l}} \hat{G}_0(l)\, \mathrm{e}^{-dl}\, \mathrm{d}l + \mathrm{e}^{-dl^*}\, \Delta F_\sigma + \mathrm{e}^{-d(l^*+\hat{l})} \hat{F}(\hat{l}). \tag{3.12}$$

The free energy F (3.12) is easily calculated in terms of the coupling parameters r, u and H of the Hamiltonian $\mathscr{H}$ (2.1). To this end the matching conditions $r_L(l^*) = \mathrm{O}(1)$ and $\hat{r}(\hat{l}) = \mathrm{O}(1)$ have to be evaluated. However, we focus the attention on the magnetic properties here, leaving the caloric properties for a separate paper.

The susceptibilities and the equation of state follow from the transformation (2.8) of the magnetization. The magnetization at the matching point $M(l^*)$ has to be calculated from the condition $\langle\sigma_0^1\rangle = 0$. The usual procedure to calculate $M(l^*)$ would be a Feynman graph expansion of $\tilde{\mathscr{H}}(\sigma, \boldsymbol{S}_\perp)$ [3]. This would lead to the logarithmic terms like $\ln r_T(l^*)$ with prefactors of $\mathrm{O}(\varepsilon)$. This calculation would lead to unexponentiated results for the equation of state and for the susceptibilities very similar to the original Feynman graph expansion by Brezin *et al* [3]. In this paper I make use of the fact that $\tilde{H}(l^*)$ is the source term for σ_0^1 in the Hamiltonian $\tilde{\mathscr{H}}(l^*)$ (3.3); thus $\langle\sigma_0^1\rangle$ can be calculated as a derivative of $\tilde{F}(l^*)$ (3.4):

$$\langle\sigma\rangle = \frac{\partial \tilde{F}(l^*)}{\partial \tilde{H}(l^*)} = \frac{\partial}{\partial \tilde{H}(l^*)} (\hat{F} + \Delta F_\sigma). \tag{3.13}$$

As shown in (3.4), $\tilde{F}(l^*)$ consists of two parts: the contribution of the longitudinal modes ΔF_σ only and the free energy $\hat{F}$ of the effective Goldstone system. The derivative of ΔF_σ follows from (3.5) as

$$\frac{\partial \Delta F_\sigma}{\partial \tilde{H}(l^*)} = -\frac{\tilde{H}(l^*)}{r_{\mathrm{L}}(l^*)} + 3\frac{w_2(l^*)}{r_{\mathrm{L}}(l^*)} G_{\mathrm{L}}^{-1}(r_{\mathrm{L}}(l^*)) \tag{3.14}$$

where we have explicitly noted the dependence of G_{L} on $r_{\mathrm{L}}(l^*)$. $\hat{F}$ (3.11) depends on $\tilde{H}(l^*)$ in a simple way via $\hat{r}$, so that one obtains

$$\frac{\partial \hat{F}}{\partial \tilde{H}(l^*)} = \frac{\partial \hat{F}}{\partial \hat{r}} \frac{\partial \hat{r}}{\partial \tilde{H}(l^*)} = \hat{E}(\hat{t}, \hat{u}) 2 \frac{w_1(l^*)}{r_{\mathrm{L}}(l^*)}. \tag{3.15}$$

$\hat{E}$ is the energy [6, 12] of the effective Goldstone system, defined as the temperature derivative of the free energy

$$\hat{E} := \frac{\partial \hat{F}}{\partial \hat{r}} = -\frac{1}{2}\int_q \langle \boldsymbol{S}_{\perp_q} \cdot \boldsymbol{S}_{\perp_{-q}} \rangle = \frac{\partial \hat{F}}{\partial \hat{t}}. \tag{3.16}$$

Likewise, the specific heat of $\hat{\mathscr{H}}$ (3.6) is defined as

$$\hat{C} := \frac{\partial^2 \hat{F}}{\partial \hat{r}^2} = \frac{\partial^2 \hat{F}}{\partial \hat{t}^2}. \tag{3.17}$$

Combining equations (3.14), (3.15) with (3.13), one is led to the equation of state of the isotropic S^4-model

$$\begin{aligned} H(l^*) - r(l^*)M(l^*) - 4u(l^*)M^3(l^*) \\ = 12u(l^*)M(l^*)G_{\mathrm{L}}(r_{\mathrm{L}}(l^*)) + 8u(l^*)M(l^*)\hat{E}(\hat{t}, \hat{u}) \end{aligned} \tag{3.18}$$

where I have inserted the definition of $\tilde{H}(l)$ and $w_i(l)$ (2.10). The first term on the RHS of (3.18) is the fluctuation correction of the longitudinal modes in the Landau regime $r_{\mathrm{L}}(l^*) = \mathrm{O}(1)$. In the case of an Ising system ($n = 1$) this term is the only contribution to the equation of state [6]. The second term includes all contributions of the critical Goldstone modes to the equation of state. From (3.18) one can derive the susceptibilities χ_{L} and χ_{T} using the transformation (2.9) of the susceptibilities. The transverse susceptibility $\chi_{\mathrm{T}}(l^*)$ at the matching point l^* is simply given by

$$\chi_{\mathrm{T}}^{-1}(l^*) = \frac{H(l^*)}{M(l^*)} \tag{3.19}$$

and the longitudinal susceptibility follows by differentiation of the equation of state (3.18) as

$$\begin{aligned} \chi_{\mathrm{L}}^{-1}(l^*) &= \frac{\partial H(l^*)}{\partial M(l^*)} \\ &= r(l^*) + 12u(l^*)M^2(l^*) + 12u(l^*)G_{\mathrm{L}}(r_{\mathrm{L}}(l^*)) + 8u(l^*)\hat{E}(\hat{t}, \hat{u}) \\ &\quad + 64u^2(l^*)M^2(l^*)\hat{C}(\hat{t}, \hat{u}) \end{aligned} \tag{3.20}$$

where again (2.10) is used. I point out that the results (3.18)-(3.20) do not contain logarithmic divergences from the Goldstone modes. They are exponentiated in the energy $\hat{E}(\hat{t}, \hat{u})$ and the specific heat $\hat{C}(\hat{t}, \hat{u})$ of the effective Goldstone modes in a very natural way.

4. The susceptibilities and the equation of state

In order to extract physical results from (3.18)-(3.20) and from the transformation of $\hat{r}$ and $\hat{u}$ (3.7), (3.8), the matching parameter l^* has to be eliminated from the matching condition. Beforehand, it is sensible to introduce the abbreviations

$$\begin{aligned} T_{\rm L}(l^*) &:= t(l^*) + 12u(l^*)M^2(l^*) = r_{\rm L}(l^*) + {\rm O}(u(l^*)) \\ T_{\rm T}(l^*) &:= t(l^*) + 4u(l^*)M^2(l^*) = r_{\rm T}(l^*) + {\rm O}(u(l^*)). \end{aligned} \tag{4.1}$$

Both 'temperatures' may be interpreted as critical distances in the M-T phase diagram. $T_{\rm L}(l^*) \simeq |-2t(l^*)|$ (see (2.14)) is the renormalized temperature distance, whereas $T_{\rm T}(l^*)$ is identified as the distance from the coexistence curve using (2.14). Both quantities are related to the relevant scaling fields t (2.11) and $\hat{t}$ (3.10) respectively: below T_c the relevant scaling field is $\hat{t}$, it vanishes at the coexistence curve. Above T_c and for $H=0$, $t=(T-T_c)/T_c$ is the only relevant scaling field vanishing at T_c. Thus, the critical point $T=T_c$, $H=0$ is a bicritical point since t as well as $\hat{t}$ vanish there independently.

As stated above, the matching parameter l^* is fixed by the condition that the longitudinal fluctuations become non-critical. This is obviously satisfied by $T_{\rm L}(l^*)=1$ which is equivalent to $r_{\rm L}(l^*)=1$ up to terms of ${\rm O}(u(l^*))$.

The integral $G_{\rm L}$ (3.9) over the longitudinal propagator, which occurs in equations (3.18)-(3.20), is easily evaluated:

$$\int_q \frac{1}{(r_{\rm L}(l^*)+q^2)} = \frac{K_4}{2}(\Lambda^2 + r_{\rm L}(l^*)[\ln r_{\rm L}(l^*) - \ln(r_{\rm L}(l^*)+\Lambda^2)]). \tag{4.2}$$

Since $r_{\rm L}(l^*) = {\rm o}(1)$, this constant fluctuation correction term may be neglected as it is unimportant compared to the non-analytic terms $\hat{E}(\hat{t}, \hat{u})$ and $\hat{C}(\hat{t}, \hat{u})$ in the equation of state and the susceptibilities (3.18)-(3.20). Thus, one obtains from (3.18), (3.19)

$$\chi_{\rm T}^{-1}(l^*) = \frac{H(l^*)}{M(l^*)} = T_{\rm T}(l^*) + 8u(l^*)\hat{E}(\hat{t}, \hat{u}) \tag{4.3}$$

and the inverse longitudinal susceptibility (3.20) is evaluated as:

$$\chi_{\rm L}^{-1}(l^*) = 1 + 8u(l^*)\hat{E}(\hat{t}, \hat{u}) + 64u^2(l^*)M^2(l^*)\hat{C}(\hat{t}, \hat{u}). \tag{4.4}$$

Inverting this result, one obtains to leading order in $u(l^*)$

$$\chi_{\rm L}(l^*) = 1 - 8u(l^*)\hat{E}(\hat{t}, \hat{u}) - 8u(l^*)(1 - T_{\rm T}(l^*))\hat{C}(\hat{t}, \hat{u}) \tag{4.5}$$

where (4.1) has been used to eliminate $M(l^*)$. It will be shown below that near the coexistence curve the specific heat term dominates since it has the strongest singularity. It is interesting that this relation between the longitudinal susceptibility of an isotropic n-component spin system to the specific heat of a $(n-1)$-dimensional system has been established in the nonlinear σ-model too. The equation of state (3.18) may now be inserted into the transformation (3.7) for $\hat{r}$ to eliminate $\tilde{H}(l^*)$:

$$\hat{r} = r_T(l^*) + 4u(l^*)G_L + 64\frac{u^2(l^*)M^2(l^*)}{r_L(l^*)}\hat{E}(\hat{t}, \hat{u}) \tag{4.6}$$

which leads to the effective scaling field $\hat{t}$ (3.10):

$$\begin{aligned}\hat{t} &= T_T(l^*) + 64u^2(l^*)M^2(l^*)\hat{E}(\hat{t}, \hat{u}) \\ &= T_T(l^*) + (1 - T_T(l^*))8u(l^*)\hat{E}(\hat{t}, \hat{u}).\end{aligned} \tag{4.7}$$

The effective interaction $\hat{u}$ is given by (3.8). It depends, in principle, on momentum $\boldsymbol{q}_1 + \boldsymbol{q}_2$ via the integral over the propagator $G_L^{-1}(r_L(l^*))$. However, standard scaling arguments on irrelevant variables [13, 14] prove that the momentum-dependent part of this $S_\perp^4$-interaction is irrelevant in very much the same way as the momentum-dependent part of the S^4-operator is irrelevant in S^4-theory. The effective interaction (3.8) can be written as

$$\begin{aligned}\hat{u} &= u(l^*) - 8\frac{u^2(l^*)M^2(l^*)}{r_L(l^*)} \\ &= u(l^*) - \frac{T_L(l^*) - T_T(l^*)}{T_L(l^*)}u(l^*) + O(u^2(l^*)) \\ &= u(l^*)\frac{T_T(l^*)}{T_L(l^*)}\end{aligned} \tag{4.8}$$

if one uses the definitions (4.1) as well as the matching condition $T_L(l^*) = 1$. Equations (4.7) and (4.8) show that the effective temperature $\hat{t}$ and the interaction $\hat{u}$ depend on the relative magnitude of the distances $T_L(l^*)$ and $T_T(l^*)$ in the M–T phase diagram. Ordinary isotropic critical behaviour is characterized by $T_T(l^*) \simeq -T_L(l^*) = 1$. This case is less interesting since transverse as well as longitudinal modes become non-critical at l^*. In this limit the interaction $\hat{u}$ (4.8) is simply $u(l^*)$ and the effective temperature $\hat{t}$ (4.7) is equal to $T_L(l^*) = 1$ so that a special renormalization of $\hat{\mathscr{H}}$ as in section 3 is not necessary. The energy $\hat{E}$ and the specific heat $\hat{C}$ then simply reduce to the corresponding expressions given by Landau theory plus fluctuation corrections.

However, the renormalization of the Goldstone system performed in section 3 is designed to treat the interesting region near the coexistence curve where the transverse modes are critical. One can see from (4.8) that the interaction $\hat{u}$ among the Goldstone modes depends on the ratio of the 'transverse temperature' $T_T(l^*)$ and the 'longitudinal temperature' $T_L(l^*)$. Thus, it depends on the position of the system in the M–T plane. It vanishes at the coexistence curve since $T_T(l^*)$ goes to zero. This vanishing interaction is a well known feature of Goldstone behaviour and has been obtained by spin-wave calculations [1, 2] and in the nonlinear σ-model [4, 5]. It is satisfactory that this feature naturally appears in this treatment.

Since $\hat{u}$ vanishes at the coexistence curve it is convenient to make use of the energy $\hat{E}(\hat{t}, \hat{u})$ and the specific heat $\hat{C}(\hat{t}, \hat{u})$ in tricritical scaling fields, in order to extract the leading $\hat{t}$-singularity relevant near the coexistence curve. The crossover calculations in the disordered phase for the $(n-1)$-dimensional isotropic system $\hat{\mathscr{H}}$ (3.6) led to [6, 12]:

$$\hat{E}(\hat{t}, \hat{u}) = \frac{(n-1)K_4}{4\alpha_t}\frac{1-(1-\hat{\mu}_2)^{\alpha_c/\alpha_t}}{(\alpha_c/\alpha_t)\hat{\mu}_2}\hat{t} - \frac{(n-1)K_4}{4\alpha_t}\frac{\hat{g}_{1t}^{2-\alpha_t}}{\hat{t}}F_0(\hat{c}_t) \tag{4.9}$$

with the tricritical crossover function

$$F_0(\hat{c}_t)=\frac{(1+\hat{c}_t)^{\alpha_c/\alpha_t}-1}{(\alpha_c/\alpha_t)\hat{c}_t}. \tag{4.10}$$

Equations (4.9), (4.10) describe the crossover between the tricritical and the critical singularities inherent in the S^4-model. The first term in (4.9) is the leading regular part of the energy. The second term is the singular part written in explicit scaling form with the tricritical temperature scaling field

$$\hat{g}_{1t}=\hat{t}(1-\hat{\mu}_2)^{-\sigma_t/\phi_t} \tag{4.11}$$

and the scaling variable

$$\hat{c}_t=\hat{\mu}_2(1-\hat{\mu}_2)^{\phi_c/\phi_t}\hat{t}^{-\phi_t} \tag{4.12}$$

ϕ_c and ϕ_t are the crossover exponents of the critical and of the tricritical fixed point of the $(n-1)$-dimensional S^4-model. Since $\hat{\mu}_2=\hat{u}/\hat{u}^*=4(n+7)K_4\hat{u}/\varepsilon$ is much smaller than 1 near the coexistence curve, one may use the simplified expression

$$\hat{E}(\hat{t},\hat{u})=\frac{(n-1)K_4}{4\alpha_t}\hat{t}-\frac{(n-1)K_4}{4\alpha_t}\hat{t}^{1-\alpha_t}F_0(\hat{c}_t) \tag{4.13}$$

instead of (4.9). However, if one is interested in the crossover from Goldstone behaviour to isotropic critical behaviour, the full expression (4.9) has to be used.

Inserting the energy $\hat{E}$ into the equation for the effective temperature $\hat{t}$ (4.7) one realizes that the singular term $\hat{t}^{1-\alpha_t}$ with the tricritical exponent of the specific heat $\alpha_t=\varepsilon/2$ dominates for $T_T(l^*)\ll 1$. This becomes evident when one rewrites (4.7) in the form

$$\hat{t}+T_T(l^*)8u(l^*)\hat{E}(\hat{t},\hat{u})=T_T(l^*)+8u(l^*)\hat{E}(\hat{t},\hat{u}). \tag{4.14}$$

The terms on the RHS are of $O(\hat{t}^{1-\alpha_t})$ whereas those on the LHS are of $O(\hat{t},\hat{t}^{2-2\alpha_t})$. Near the coexistence-curve ($T_T(l^*)\ll 1$, $\hat{t}\ll 1$) (4.14) reduces to the simple relation

$$T_T(l^*)\sim\hat{t}^{1-\alpha_t}. \tag{4.15}$$

This result explicitly shows that the effective temperature $\hat{t}$ of the Goldstone system is Fisher renormalized with respect to the 'temperature' $T_T(l^*)$ of the transverse modes in the original renormalized Hamiltonian (3.1)-(3.3). I point out that the LHS of equation (4.14) is negligible for $\hat{t}\ll 1$ only, leading to $T_T(l^*)\sim\hat{t}^{1-\alpha_t}$. In order to describe the crossover to isotropic critical behaviour one has to take all terms into account.

After this discussion of the effective coupling parameters of the Goldstone system and their relation to the original coupling parameters of the Hamiltonian (3.1)-(3.3), one may eliminate the l^* dependence from the above results in the usual way [6, 7, 12]. The matching condition $T_L(l^*)=1$ can be written as

$$t(l^*)+3\frac{u(l^*)}{u^*}m^2(l^*)=1 \tag{4.16}$$

using the abbreviation $m=\sqrt{4u^*}\,M$ and $h=\sqrt{4u^*}\,H$. In order to normalize the equation of state below in a proper way, it is convenient to introduce normalization factors

$$\bar{t}=at \qquad \bar{m}=\sqrt{b}\,m \qquad \bar{h}=ch. \tag{4.17}$$

Inserting the solutions $t(l^*)$ and $u(l^*)$ (2.10) and the transformation of the magnetization $m(l^*)$ (2.8) into (4.16), one obtains

$$\frac{1}{a}\frac{g_{1c}\exp(y_{1c}l^*)}{Z^{\sigma_c/\phi_c}}+\frac{3}{b}\frac{m^2\exp(2y_m l^*)}{Z}=1 \tag{4.18}$$

with

$$Z=1+g_{2c}\exp(y_{2c}l^*) \tag{4.19}$$

where I have used the nonlinear critical scaling fields introduced by Riedel and Wegner [15]:

$$g_{1c}=t\left(\frac{u}{u^*}\right)^{-\sigma_c/\phi_c} \qquad g_{2c}=\left(1-\frac{u}{u^*}\right)^*\frac{u^*}{u}. \tag{4.20}$$

y_{1c}, y_{2c} are the exponents of these scaling fields, given by

$$y_{1c}=2-\frac{A}{B}\varepsilon \qquad y_{2c}=-\varepsilon \qquad y_m=2-\varepsilon \tag{4.21}$$

and y_m is the exponent of m^2. It has been shown in previous works that the function $\tilde{L}=e^{l^*}$ shows scaling behaviour [6, 12]. $\tilde{L}$ can be written in the form

$$\tilde{L}(g_{1c}, m, g_{2c})=m^{-1/y_m}L(\xi_c, c_m) \tag{4.22}$$

with the scaling variables

$$\xi_c=g_{1c}m^{-y_{1c}/y_m}=g_{1c}m^{-1/\beta_c} \qquad c_m=g_{2c}m^{y_{2c}/y_m}=g_{2c}m^{\phi_c/\beta_c}. \tag{4.23}$$

From (4.18) one obtains the equation for the scaling function L, defined in (4.22):

$$\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{3}{b}\frac{L^{2y_m}}{Z}=1 \tag{4.24}$$

with $Z=1+c_m L^{y_{2c}}$. Note that $Z=1$ for a system at the fixed point $u=u^*$. Equation (4.24) is a transcendental equation for $L(\xi_c, c_m)$ which has to be solved numerically if one is interested in the complete crossover. An analytical discussion is possible in the tricritical and critical limit and in some regions of the $M-T$ phase diagram. $T_T(l^*)$ (4.1) is obtained performing the same calculations as for $T_L(l^*)$ above:

$$T_T(l^*)=\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}. \tag{4.25}$$

Combining this result with equation (4.14), one obtains the effective temperature $\hat{t}$ in terms of the scaling variables ξ_c and c_m:

$$\hat{t}+\left[\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}\right]8\frac{u^*}{Z}\hat{E}=\left[\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}\right]+8\frac{u^*}{Z}\hat{E}. \tag{4.26}$$

The fixed point value u^* appears in (4.26) because of the normalization $\mu_2=u/u^*$. Equation (4.26) is a transcendental equation for $\hat{t}(\xi_c, c_m)$ which must be solved numerically in general. It describes the dependence of the effective temperature of the Goldstone system on the scaling variables ξ_c and c_m. The effective coupling $\hat{\mu}_2$ follows from (4.8) as

$$\hat{\mu}_2=\frac{n+7}{n+8}\frac{1}{Z}\left[\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}\right]. \tag{4.27}$$

This equation shows that the effective coupling depends on the scaling variables ξ_c and c_m (4.23). Equations (4.9), (4.10) show that the energy $\hat{E}$ depends on the crossover variable $\hat{c}_t$. This variable has to be calculated from $\hat{\mu}_2$ (4.27) and $\hat{t}$ (4.26), if one is interested in the crossover from coexistence curve behaviour to isotropic critical behaviour.

Performing the same manipulations as above with equations (4.3) and (4.5), one obtains the general crossover results for the equation of state and the susceptibilities:

$$\frac{h}{m^{\delta_c}}=\frac{c}{\sqrt{b}}L^{-2}\left\{\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}+8\frac{u^*}{Z}\hat{E}(\hat{t},\hat{u})\right\} \tag{4.28}$$

$$\frac{\chi_T}{m^{1-\delta_c}}=L^2\left\{\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}+\frac{1}{b}\frac{L^{2y_m}}{Z}+8\frac{u^*}{Z}\hat{E}(\hat{t},\hat{u})\right\}^{-1} \tag{4.29}$$

$$\frac{\chi_L}{m^{1-\delta_c}}=L^2\left\{1-8\frac{u^*}{Z}\hat{E}(\hat{t},\hat{u})-8\frac{u^*}{Z}\frac{\hat{E}(\hat{t},\hat{u})}{\hat{t}}\left[1-\frac{1}{a}\frac{\xi_c L^{y_{1c}}}{Z^{\sigma_c/\phi_c}}-\frac{1}{b}\frac{L^{2y_m}}{Z}\right]\right\}. \tag{4.30}$$

In the last equation I have used the well known relation $\hat{C}=\hat{E}/\hat{t}$ [6, 12]. The complexity of (4.28)-(4.30) together with the transcendental equations for L (4.24) and $\hat{t}$ (4.26) does not make these results very attractive. However, one has to recall that these equations describe the critical behaviour of a system with arbitrary S^4-coupling u in the whole $M-T$ phase diagram.

5. Coexistence behaviour

In order to discuss the behaviour near the coexistence-curve, it is convenient to restrict the analysis to the case $u=u^*$. In this case the system (2.1) is at the Heisenberg fixed point and the crossover with respect to u is eliminated. This is the situation described in previous papers [3-10] on the subject.

Near the coexistence curve the results (4.24)-(4.30) simplify since only the leading powers of $\hat{t}$ are important for $\hat{t}\ll 1$. Inserting the energy $\hat{E}$ (4.13) into the equation for the effective temperature $\hat{t}$ (4.14), one gets

$$T_T(l^*)=\left[\frac{1}{a}\xi_c L^{y_{1c}}+\frac{1}{b}L^{2y_m}\right]=\frac{\frac{9}{n+8}\hat{t}+\frac{n-1}{n+8}\hat{t}^{1-\alpha_t}F_0(\hat{c}_t)}{1-\frac{n-1}{n+8}\hat{t}+\frac{n-1}{n+8}\hat{t}^{1-\alpha_t}F_0(\hat{c}_t)}. \tag{5.1}$$

For $\hat{t}\ll 1$ the denominator can be ignored and the leading term is the singular term of the energy. Thus, for $\hat{t}\ll 1$:

$$T_T(l^*)=\left[\frac{1}{a}\xi_c L^{y_{1c}}+\frac{1}{b}L^{2y_m}\right]=\frac{9}{n+8}\hat{t}+\frac{n-1}{n+8}\hat{t}^{1-\alpha_t}F_0(\hat{c}_t) \tag{5.2}$$

up to terms of order $\hat{t}^{2-\alpha_t}$. This equation explicitly shows that the critical distance $T_T(l^*)$ (4.1) is related to the temperature $\hat{t}$ of the Goldstone system (3.6) by a Fisher renormalization. The physical origin for this Fisher renormalization is the coupling of the critically fluctuating transverse modes to the non-critical longitudinal modes in the Goldstone regime. The general results (4.28)-(4.30) become very simple now. Inserting the energy $\hat{E}$ (4.13) as well as (5.2) into (4.28), one is led to the equation of state at the coexistence curve

$$\frac{h}{m^{\delta_c}}=\frac{c}{\sqrt{b}}L^{-2}\hat{t}. \tag{5.3}$$

The transverse susceptibility (4.29) is given by

$$\frac{\chi_T}{m^{1-\delta_c}}=L^2\hat{t}^{-1} \tag{5.4}$$

and the longitudinal susceptibility (4.30) is evaluated in the same way observing that the term $\hat{E}/\hat{t}$, i.e. the specific heat $\hat{C}\simeq\hat{t}^{-\alpha_t}F_0(\hat{c}_t)$ at the Gaussian fixed point, dominates for $\hat{t}\ll 1$:

$$\frac{\chi_L}{m^{1-\delta_c}}=L^2\left[\frac{9}{n+8}+\frac{n-1}{n+8}\hat{t}^{-\alpha_t}F_0(\hat{c}_t)\right]. \tag{5.5}$$

This result can be nicely compared with the result of the nonlinear σ-model [4, 5]. In these works it has been shown that the fluctuations of the longitudinal mode χ_L are related to the fluctuations of the transverse modes at their trivial fixed point by

$$\chi_L=\int d^dx\left\langle\sum_{i=2}^{n}S_i^2(x)\sum S_i^2(0)\right\rangle. \tag{5.6}$$

Since the RHS of (5.6) is the specific heat of an $(n-1)$-dimensional spin system, equation (5.5) perfectly agrees with the result of the nonlinear σ-model.

The equation of state and the transverse susceptibility near the coexistence curve have a very simple structure too. They correspond to a system with temperature scaling field $\hat{t}$ and vanishing interaction $\hat{u}$. The result that the interaction $\hat{u}$ vanishes at the coexistence curve is verified directly from (4.27), inserting the leading singular term of (5.2) into (4.27):

$$\hat{\mu}_2=\frac{n+7}{n+8}\left[\frac{\sqrt{b}}{c}L^2\right]^{1-\alpha_t}\frac{n-1}{n+8}\left[\frac{h}{m^{\delta_c}}\right]^{1-\alpha_t} \tag{5.7}$$

where the equation of state (5.3) has been used to replace $\hat{t}$. Equation (5.7) explicitly shows that the coexistence curve is governed by a trivial Gaussian fixed point where the $S_\perp^4$-interaction $\hat{\mu}_2$ vanishes for $h\to 0$. Hence, the critical dimension is determined by the $S_\perp^6$-operator to give $d_c=3$ as in tricritical phenomena. Thus, the results concerning coexistence behaviour derived in this work are exact for $d\geqslant 3$ for all spin dimensions n. This feature compares well with the works on the nonlinear σ-model, which arrive at the same result [4, 5]. A coexistence fixed point of Gaussian character has also been found by Lawrie in his RG analysis of the S^4-model [10]. He also concluded that his results concerning the coexistence curve should be exact. However, with his renormalization scheme he did not discover the mechanism of Fisher renormalization of Goldstone behaviour by the longitudinal modes.

The above results (5.3)-(5.5) can be cast into the usual form by an appropriate normalization according to (4.17). The crossover variable $\hat{c}_t$ (4.12) which enters the crossover function F_0 in the above results is given by

$$\hat{c}_t=\frac{n+7}{n+8}\frac{n-1}{n+8}\left[\frac{\sqrt{b}}{c}L^2\frac{h}{m^{\delta_c}}\right]^{1-2\alpha_t}. \tag{5.8}$$

The equation for the scaling function L (4.22) assumes the simpler form

$$\frac{1}{a}\,\xi_c L^{y_{1c}}+\frac{3}{b}\,L^{y_{2m}}=1 \tag{5.9}$$

since $Z=1$ for $u=u^*$. Thus, the equation of state in the Widom scaling form is obtained from (5.2) and (5.3) as

$$\frac{h}{m^{\delta_c}}=\frac{c}{\sqrt{b}}\,L^{-2}\left[\frac{n+8}{n-1}\,F_0^{-1}\right]^{1/(1-\alpha_t)}\left[\frac{1}{a}\,\xi_c L^{y_{1c}}+\frac{1}{b}\,L^{2y_m}\right]^{1/(1-\alpha_t)} \tag{5.10}$$

where $F_0=O(1)$ is the limit $h\to 0$ of the crossover function $F_0(\hat{c}_t)$. One may now choose the normalization (4.17) in such a way that $f(\xi_c=-1)=0$ and $f(\xi_c=0)=1$. In addition, I choose $L(\xi_c=-1)=1$ so that the scaling function L (5.9) is unity on the coexistence curve:

$$a=2 \qquad b=2 \qquad c=\sqrt{2}\left(\frac{2}{3}\right)^{1/y_m}\left[3\,\frac{n-1}{n+8}\,F_0\right]^{1/(1-\alpha_t)}. \tag{5.11}$$

With these normalizations one obtains

$$\frac{h}{m^{\delta_c}}=\left[\frac{t}{m^{1/\beta_c}}+1\right]^{1/(1-\alpha_t)} \tag{5.12}$$

as the equation of state near the coexistence curve. Likewise, the transverse susceptibility follows from (5.4) as

$$\frac{\chi_T^{-1}}{m^{\delta_c-1}}=\frac{1}{\sqrt{2}}\,\frac{h}{m^{\delta_c}}=\frac{1}{\sqrt{2}}\left[\frac{t}{m^{1/\beta_c}}+1\right]^{1/(1-\alpha_t)} \tag{5.13}$$

and the longitudinal susceptibility follows from (5.5) as

$$\frac{\chi_L}{m^{1-\delta_c}}=L^2\left[\frac{9}{n+8}+\frac{n-1}{n+8}\,F_0\,\frac{1}{\sqrt{2}}\,L^{-2\alpha_t}\left(\frac{h}{m^{\delta_c}}\right)^{-\alpha_t}\right] \tag{5.14}$$

so that one obtains

$$\frac{\chi_L}{m^{1-\delta_c}}\sim\left(\frac{h}{m^{\delta_c}}\right)^{-\alpha_t} \tag{5.15}$$

near the coexistence curve. Equation (5.14) may also be expressed as

$$\begin{aligned}\frac{\chi_L}{m^{1-\delta_c}}&=L^2\left[\frac{9}{n+8}+\left(\frac{n-1}{n+8}\,F_0\right)^{\alpha_t/(1-\alpha_t)}\left(\frac{1}{2}\,\xi_c L^{y_{1c}}+\frac{1}{2}\,L^{2y_m}\right)^{\alpha_t/(1-\alpha_t)}\right]\\&=L^2\left[\frac{9}{n+8}+\frac{n-1}{n+8}\,F_0 L^{-2\alpha_t}\left(\frac{\chi_T}{m^{1-\delta_c}}\right)^{\alpha_t}\right].\end{aligned} \tag{5.16}$$

As mentioned above, the results (5.12), (5.13), (5.15) for the functional form of the equation of state and the susceptibilities near the coexistence curve do not depend on the spin dimension n and are exactly given by the specific heat exponent $\alpha_t=\frac{1}{2}$ for $d\geq 3$. The functional dependence (5.12) $y=(x+1)^t$ of the scaled magnetic field $y=hm^{-\delta_c}$ on the scaled magnetization $x=tm^{-1/\beta}$ has been conjectured in early RG works on the S^4-model [2, 3] and has been confirmed by works on the nonlinear σ-model [4, 5]. Lawrie has confirmed this result within the S^4-model in the spherical limit $n-\infty$. Equation (5.12) verifies this result for arbitrary spin dimension n and shows that $\iota = 1/(1-\alpha_\iota) = 2$ for $d \geqslant 3$ which is in perfect agreement with the nonlinear σ-model. However, for $d = 3$ one may include logarithmic corrections if necessary. The longitudinal susceptibility (5.14) has the expected form too. The dominating term near the coexistence curve is the non-analytic one which diverges as $H^{-1/2}$ for $d \geqslant 3$ (5.15), in accord with the nonlinear σ-model. One may derive from (5.14) a condition for the observation of Goldstone singularities. The Goldstone term in (5.14) has to be distinctly larger than the constant term which describes the isotropic critical behaviour near T_c. This leads to the condition

$$\frac{h}{m^{\delta_c}} < \frac{1}{2}\left(\frac{n-1}{9}\right)^2 = \begin{cases} 0.006 & \text{for } n = 2 \\ 0.024 & \text{for } n = 3. \end{cases} \tag{5.17}$$

It is obvious that a proper identification of Goldstone singularities is possible only in a small regime around the coexistence curve. Measurements in this regime are difficult since domain effects become important. Furthermore, anisotropy in real ferromagnets and dipolar interactions have to be included in the theoretical description. The method presented in this paper seems to be well suited to treat these problems.

References

[1] Vaks V G, Larkin A I and Pikin S A 1968 *Sov. Phys.-JETP* **26** 188
[2] Vaks V G, Larkin A I and Pikin S A 1968 *Sov. Phys.-JETP* **26** 647
[3] Brezin E, Wallace D J and Wilson K G 1973 *Phys. Rev.* B **7** 232
[4] Brezin E and Wallace D J 1973 *Phys. Rev.* B **7** 1967
[5] Brezin E and Zinn-Justin J 1976 *Phys. Rev.* B **14** 3110
[6] Rudnick J and Nelson D R 1976 *Phys. Rev.* B **13** 2208
[7] Nelson D R 1976 *Phys. Rev.* B **13** 2222
[8] Schäfer L and Horner H 1978 *Z. Phys.* B **29** 251
[9] Nicoll J F and Chang T S 1978 *Phys. Rev.* A **17** 2083
[10] Lawrie I D 1981 *J. Phys. A: Math. Gen.* **14** 2489
[11] Fisher M E 1968 *Phys. Rev.* **176** 257
[12] Heuer H-O and Wagner D 1989 *Phys. Rev.* B **40** 2502
[13] Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75
[14] Bruce A D, Droz M and Aharony A 1976 *J. Phys. C: Solid State Phys.* **9** 825
[15] Riedel E K and Wegner F J 1974 *Phys. Rev.* B **9** 294